Exploring Polynomial Sequence Formulas using Cramer's Rule Approach

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***Abstract*—Mathematics and its different branches, from geometry to linear algebra, are vital to daily life, where many tools have been developed through rigorous observation of patterns. Patterns are found in sequences such as the arithmetic and Fibonacci sequences. However, at first glance, especially for students, polynomial sequences may not exhibit a clear pattern, as they do not always share a common first difference or ratio between terms. In addition, the process of solving these sequences tends to be intensive and tedious. Therefore, the goal of this study is to construct a general formula for the n[th] degree polynomial sequence. After utilizing concepts from linear algebra, the following equation was derived:**

$$f(x) = \sum_{i=0}^{n} \frac{\det(A_{B(i+1)})}{\det(A)} x^{n-i}$$

The formula provides a way to solve for the coefficients using determinants of matrices. For practical applications, $\{n \in \mathbb{Z} \mid n \in (2,3)\}$ are recommended as they involve determinants of manageable matrices. For computerized processes, the general formula may be coded as an alternative. However, since determinant calculations can be memory-intensive, algorithms such as LU decomposition and the Strassen algorithm, which have lower time complexities, are recommended.

***Keywords*—Polynomial Sequence, Cramer's Rule, Determinant, Matrices, Strong Induction**

I. INTRODUCTION

Mathematics, from the basics of arithmetic to higher-level concepts, has enabled civilizations to flourish and develop over the millennia.

Many mathematicians have developed concepts that touch upon many branches of mathematics through rigorous observation [1]. One of the main ways mathematical concepts were developed was through finding patterns. This is because problems once thought to be complex were simplified which revealed connections that could span different branches of mathematics, which ultimately lead to elegant solutions [2]. In intermediate algebra, applications of patterns are seen through sequences such as arithmetic sequences and Fibonacci sequences [3].

The explicit formula of the Fibonacci sequence is as follows:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \quad (1)$$

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}} \quad (2)$$

The value of φ was identified by Johannes Kepler as its converging ratio, which is shown by the limit as follows:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2} \quad (3)$$

Sequences have applications that are evident in many fields such as finance, engineering, population growth, and physical measurements [4]. However, decoding these sequences may not be straightforward.

Polynomial sequences operate using a general formula where the degree of the equation that describes it is generally in the form $n > 1, n \in \mathbb{Z}^+$. Quadratic sequences describe sequences whose degree is 2. It is commonly taught in high school mathematics as it is intuitive [5]. The general equation for a quadratic and cubic sequence is as follows:

$$f(x) = ax^2 + bx + c \quad (4)$$

$$f(x) = ax^3 + bx^2 + cx + d \quad (5)$$

In these equations, $a, b, c, and\ d$ are coefficients and x is the variable. In general, an n[th] degree polynomial sequence is described by the general polynomial equation:

$$f(x) = a_1x^n + a_2x^{n-1} + a_3x^{n-2} + a_4x^{n-3} + \cdots + a_nx + a_{n+1} \quad (6)$$

However, unlike an arithmetic sequence where the given terms may be substituted into its formula, higher-order polynomial sequences tend to be more difficult to solve as they require additional, tedious steps. An analysis of graduates solving them has shown that there are multiple solution methods for the same type of sequence [6].

A common way to solve for the general equation of a given set of terms with the degree of the equation known is by instead solving for the coefficients of the said equation.

$$f(n_1) = an_1^2 + bn_1 + c \quad (7)$$

$$f(n_2) = an_2^2 + bn_2 + c \quad (8)$$

$$f(n_3) = an_3^2 + bn_3 + c \quad (9)$$

Students in eighth grade in the Philippines are expected to be able to solve systems of equations through substitution and elimination, while non-linear systems are expected to be solved by senior high school students taking Precalculus classes [7]. In higher levels, methods such as Gaussian elimination are taught [8].

Physical tools that are made specifically to help students in solving sequences have been proposed and developed

recently. The device developed by Joson & Vargas [9], PROPlexis, is a scientific calculator-inspired tool that solves for the equations that describe a given set of terms and even graph them. However, the formulas that PROPlexis makes use of formulas that only goes up to quartic sequences ($n \leq 4$), and in general, there does not exist a closed-form general formula for an nth degree polynomial sequence. Therefore, it is the goal of this study to derive and prove such formula by applying concepts from different branches of mathematics to prove the validity of the formula.

II. METHODOLOGY

A. Derivation

In deriving the general formula for an nth degree polynomial sequence, logical deduction shall be used. With this, matrices will be used.

To reiterate, when given a polynomial equation of the form:

$$f(x) = a_1x^n + a_2x^{n-1} + a_3x^{n-2} + a_4x^{n-3} + \cdots + a_nx + a_{n+1} \quad (6)$$

Or when given a set of ordered pairs, the equation can be decoded by calculating the coefficients of each term. This is a consequence of the Invertible Matrix Theorem which states that an $n \times n$ matrix is invertible if and only if it satisfies equivalent conditions such as being full rank. The connection between matrices and these linear equations comes from how efficient the coefficients of the unknowns can be mapped with the help of matrix multiplication.

1) Proof 1

To show that the system of equations needed is equal to the number of unknowns, the Invertible Matrix Theorem shall be used.

$$f(x) = a_1x^n + a_2x^{n-1} + a_3x^{n-2} + a_4x^{n-3} + \cdots + a_nx + a_{n+1} \quad (6)$$

The goal is to create A^{-1} such that $AA^{-1} = A^{-1}A = I$, where I is the identity matrix with the form:

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad (10)$$

To create an invertible matrix, the matrix itself must be a square matrix. Therefore, it will first be assumed that this is the case by substituting different x values that are of the form x_i.

$$f(x_1) = a_1x_1^n + a_2x_1^{n-1} + \cdots + a_nx_1 + a_{n+1} \quad (11)$$

$$f(x_2) = a_1x_2^n + a_2x_2^{n-1} + \cdots + a_nx_2 + a_{n+1} \quad (12)$$

$$f(x_3) = a_1x_3^n + a_2x_3^{n-1} + \cdots + a_nx_3 + a_{n+1} \quad (13)$$

$$\vdots$$

$$f(x_{n+1}) = a_1x_{n+1}^n + a_2x_{n+1}^{n-1} + \cdots + a_nx_{n+1} + a_{n+1} \quad (14)$$

The equations remain to be linear as the substituted *x* values raised to their exponents are treated as the coefficients, while the variables of the form a_i serve as the unknowns. With this, given the vector form, $A\vec{x} = \vec{b}$:

$$A = \begin{bmatrix} x_1^n & x_1^{n-1} & \cdots & x_1 & 1 \\ x_2^n & x_2^{n-1} & \cdots & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^n & x_n^{n-1} & \cdots & x_n & 1 \\ x_{n+1}^n & x_{n+1}^{n-1} & \cdots & x_{n+1} & 1 \end{bmatrix} \quad (15)$$

Firstly, A is a square matrix and has full rank. Although the conditions for an invertible matrix are flexible, it must be proved through at least one another way since the former is an outcome of the initial assumption and to avoid contradictions.

The other way is to show that $A\vec{x} = 0$ only has $\vec{x} = 0$ has its solution. Given the vector form, $A\vec{x} = 0$,

$$\begin{bmatrix} x_1^n & x_1^{n-1} & \cdots & x_1 & 1 \\ x_2^n & x_2^{n-1} & \cdots & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^n & x_n^{n-1} & \cdots & x_n & 1 \\ x_{n+1}^n & x_{n+1}^{n-1} & \cdots & x_{n+1} & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ a_{n+1} \end{bmatrix} = 0 \quad (16)$$

It can be easily shown that $\vec{x} = 0$ as for row 1, it is represented by the equation below:

$$0 = 0x_1^n + 0x_1^{n-1} + 0x_1^{n-2} + 0x_1^{n-3} + \cdots 0x_1 + 0 \quad (18)$$

This equality continues down each row as each succeeding row consists of the same unknowns. Therefore, having no contradictions, matrix *A* must indeed be a square matrix and is invertible.

$$\vec{x} = A^{-1}\vec{b} \quad (19)$$

Since matrix *A* has been shown to be invertible, by solving for $\vec{x}$ in the equation above, it means that the system of equations has only one solution – the set of coefficients that build the desired equation, showing that the system of equations needed is equal to the number of unknowns. ■

A further application of this is Cramer's Rule, which is applicable if $det(A) \neq 0$ – one of the requirements for a matrix to be invertible, which has already been shown. Cramer's Rule provides a method for solving the unknowns of a system of linear equations using determinants [10]. It is defined as:

$$x_i = \frac{det(A_i)}{det(A)} \quad (20)$$

2) Proof 2

To prove the validity of Cramer's Rule, the multilinearity property of determinants shall be used, with $A\vec{x} = \vec{b}$. Where A is an $n \times n$ matrix, $\vec{x}$ is the matrix of unknowns, and $\vec{b}$ is the cross-multiplication product of

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n \quad (21)$$

In addition, Matrix A may be expressed as the following:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix} = [\vec{a}_1, \cdots, \vec{a}_n] \quad (22)$$

Replacing the i[th] column of Matrix A with $\vec{b}$,

$$A_i = [\vec{a}_1, \cdots, \vec{a}_{i-1}, \vec{a}_i, \vec{a}_{i+1}, \cdots, \vec{a}_n] \quad (23)$$

$$det(A_i) = det[\vec{a}_1, \cdots, \vec{a}_{i-1}, \vec{b}, \vec{a}_{i+1}, \cdots, \vec{a}_n] \quad (24)$$

$$det(A_i) = det\left[\vec{a}_1, \cdots, \vec{a}_{i-1}, \sum_{j=1}^{n} x_j \vec{a}_j, \vec{a}_{i+1}, \cdots, \vec{a}_n\right] \quad (25)$$

Since the determinant of the square matrix is linear in each column, the summation may be taken out of the determinant.

$$det(A_i) = \sum_{j=1}^{n} x_j \, det[\vec{a}_1, \cdots, \vec{a}_{i-1}, \vec{a}_j, \vec{a}_{i+1}, \cdots, \vec{a}_n] \quad (26)$$

In the summands, one is where $i = j$, which is the original matrix A. The other is when $i \neq j$, which leads to two identical columns, making the determinants of these succeeding summands be 0. Therefore,

$$\det(A_i) = x_{i=j} \det(A) \Longrightarrow det(A_i) = x_i det(A) \quad (27)$$

$$x_i = \frac{\det(A_i)}{\det(A)} \blacksquare \quad (28)$$

The definition of a determinant is given recursively by the Laplace expansion or the cofactor expansion, named after Pierre-Simon, Marquis de Laplace, its inventor, which is as follows:

$$det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \cdot \det(M_{ij}) \quad (29)$$

Table 1. Parameters for the Laplace Expansion

Symbol	Placeholder of
i	Row of Matrix M
j	Column of Matrix M
n	Size of square matrix
a	Element in Matrix M
M	Matrix where the determinant will be computed

The Laplace expansion is one of many ways to solve for the determinant of a given square matrix.

III. Result and Discussion

Upon application of the previously mentioned concepts, the following general formula for an n[th] degree polynomial sequence was derived:

$$f(x) = \sum_{i=0}^{n} \frac{det(A_{B(i+1)})}{det(A)} x^{n-i} \quad (30)$$

A. Derivation

The derivation of the formula is as follows, starting with the general equation of a polynomial:

$$f(x) = a_1 x^n + a_2 x^{n-1} + a_3 x^{n-2} + a_4 x^{n-3} + \cdots + a_n x + a_{n+1} \quad (6)$$

To find the coefficients of the equation, substitute n+1 number of values for x, of which shall be represented as $[x_1, x_2, \cdots]$. These are expressed as follows:

$$f(x_1) = a_1 x_1^n + a_2 x_1^{n-1} + \cdots + a_n x_1 + a_{n+1} \quad (11)$$

$$f(x_2) = a_1 x_2^n + a_2 x_2^{n-1} + \cdots + a_n x_2 + a_{n+1} \quad (12)$$

$$f(x_3) = a_1 x_3^n + a_2 x_3^{n-1} + \cdots + a_n x_3 + a_{n+1} \quad (13)$$

$$\vdots$$

$$f(x_{n+1}) = a_1 x_{n+1}^n + a_2 x_{n+1}^{n-1} + \cdots + a_n x_{n+1} + a_{n+1} \quad (14)$$

Let the right side of the equations be matrix A by treating the substituted x values as the coefficients, and the left side be matrix B.

$$A = \begin{bmatrix} x_1^n & x_1^{n-1} & \cdots & x_1 & 1 \\ x_2^n & x_2^{n-1} & \cdots & x_2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^n & x_n^{n-1} & \cdots & x_n & 1 \\ x_{n+1}^n & x_{n+1}^{n-1} & \cdots & x_{n+1} & 1 \end{bmatrix} \quad (15)$$

$$B = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \\ f(x_{n+1}) \end{bmatrix} \quad (31)$$

Given this, we can express the leading coefficient of the required equation by Cramer's Rule which is as follows:

$$a_1 = \frac{de\,t(A_{B1})}{de\,t(A)} \quad (32)$$

Cramer's Rule is a formula used to solve for the zeroes of a system of linear equations. However, since the coefficients are the unknowns, the *x* values given shall act as the coefficients. The succeeding coefficients along with the constant term can also be expressed in this way by modifying which column to change. Case in point:

$$a_2 = \frac{de\,t(A_{B2})}{de\,t(A)} \quad (33)$$

$$a_3 = \frac{de\,t(A_{B3})}{de\,t(A)} \quad (34)$$

$$\vdots$$

$$a_{n+1} = \frac{de\,t(A_{B(n+1)})}{de\,t(A)} \quad (35)$$

These expressions can be substituted back into the respective variables of the general polynomial equation to arrive at:

$$f(x) = \frac{det(A_{B1})}{det(A)}x^n + \frac{det(A_{B2})}{det(A)}x^{n-1} + \cdots + \frac{det(A_{Bn})}{det(A)}x + \frac{det(A_{B(n+1)})}{det(A)} \quad (36)$$

The above expression can be written in a compact form which is as follows:

$$f(x) = \sum_{i=0}^{n} \frac{det(A_{B(i+1)})}{det(A)}x^{n-i} \quad (30)$$

1) Proof by Strong Induction

Before proceeding with strong induction, Cramer's Rule is invoked. The proof of this rule was shown in the second proof of the methodology of this paper.

The principle of strong mathematical induction can be thought of as the logical, implication-based counterpart of the composition fallacy, mainly used in proving statements or formulas involving positive integers. The first step in doing so is to show that it is true for the base case. For this derivation, *n = 1*.

$$f(x) = a_1x^n + a_2x^{n-1} + a_3x^{n-2} + a_4x^{n-3} + \cdots + a_{n+1} = \sum_{i=0}^{n} \frac{det(A_{B(i+1)})}{det(A)}x^{n-i} \quad (37)$$

By letting n = 1,

$$f(x) = \sum_{i=0}^{1} \frac{det(A_{B(i+1)})}{det(A)}x^{1-i} \quad (38)$$

$$f(x) = \frac{\det(A_{B1})}{\det(A)}x + \frac{\det(A_{B2})}{\det(A)} \quad (39)$$

The equation above must also therefore be equal to the linear expression $a_nx + a_{n+1}$ on the LHS. Using this, the equation above becomes:

$$a_1x + a_2 = \frac{det(A_{B1})}{det(A)}x + \frac{det(A_{B2})}{det(A)} \quad (40)$$

Consequently,

$$a_1x - \frac{det(A_{B1})}{det(A)}x = \frac{det(A_{B2})}{det(A)} - a_2 \quad (41)$$

$$x\left(a_1 - \frac{det(A_{B1})}{det(A)}\right) = \frac{det(A_{B2})}{det(A)} - a_2 \quad (42)$$

For the equation to be equal, both sides must be 0. The only way in which this is achieved is if

$$a_1 = \frac{det(A_{B1})}{det(A)}; \frac{det(A_{B2})}{det(A)} = a_2 \quad (43)$$

Therefore, solving first for the value of a_1 :

$$a_1 = \frac{det(A_{B1})}{det(A)} \Longrightarrow det(A)a_1 = det(A_{B1}) \quad (44)$$

From the derivation of Cramer's rule, the rightmost equation is seen for $i = 1$. Therefore, the leftmost equation is true. Similarly, for:

$$a_2 = \frac{det(A_{B2})}{det(A)} \Longrightarrow det(A)a_2 = det(A_{B2}) \quad (45)$$

The rightmost equation is shown for i = 2. Thus, the determinant equivalence of a_2 must also be true. Therefore,

$$a_1x + a_2 = a_1x + a_2 \quad (46)$$

Now that the formula has been shown to be true for the basic case, for the inductive step, assume that for all permitted values of n before or equal to k where m is in between, the formula also holds true $(n \leq m \leq k)$. With this, it shall be shown that the formula also holds true for *k+1*.

To use this assumption, the following equations are shown:

By letting n = k + 1, the sum becomes:

$$f(x) = a_1x^n + a_2x^{n-1} + a_3x^{n-2} + a_4x^{n-3} + \cdots + a_{n+1} = \sum_{i=0}^{n} \frac{det(A_{B(i+1)})}{det(A)}x^{n-i} \quad (37)$$

$$f(x) = \sum_{i=0}^{k+1} \frac{det(A_{B(i+1)})}{det(A)}x^{k+1-i} \quad (47)$$

For the LHS:

$$a_1x^{k+1} + a_2x^k + a_3x^{k-1} + \cdots + a_{k+1}x + a_{k+2} \quad (48)$$

For the RHS:

$$\frac{det(A_{B1})}{det(A)}x^{k+1} + \frac{det(A_{B2})}{det(A)}x^k + \frac{det(A_{B3})}{det(A)}x^{k-1} + \frac{det(A_{B4})}{det(A)}x^{k-2} + \cdots + \frac{det(A_{B(k+1)})}{det(A)}x + \frac{det(A_{B(k+2)})}{det(A)} \quad (49)$$

Therefore,

$$a_1x^{k+1} + a_2x^k + a_3x^{k-1} + \cdots + a_{k+1}x + a_{k+2} = \frac{det(A_{B1})}{det(A)}x^{k+1} + \frac{det(A_{B2})}{det(A)}x^k + \frac{det(A_{B3})}{det(A)}x^{k-1} + \frac{det(A_{B4})}{det(A)}x^{k-2} + \cdots + \frac{det(A_{B(k+1)})}{det(A)}x + \frac{det(A_{B(k+2)})}{det(A)} \quad (50)$$

From the equation shown, the equality of each term from both sides can be shown through the following:

$$x^{k+1}\left(a_1 - \frac{det(A_{B1})}{det(A)}\right) + x^k\left(a_2 - \frac{det(A_{B2})}{det(A)}\right) + \cdots + x\left(a_{k+1} - \frac{det(A_{B(k+1)})}{det(A)}\right) = \frac{det(A_{B(k+2)})}{det(A)} - a_{k+2} \quad (51)$$

For the equation to be equal, both sides must be 0, Therefore, the equation implies that the leading coefficient satisfies:

$$a_1 = \frac{det(A_{B1})}{det(A)} \Longrightarrow det(A)a_1 = det(A_{B1}) \quad (47)$$

The rightmost equation is a segment of the derivation of Cramer's Rule for $i = 1$. Therefore, the equivalent expression of a_1 must be valid. And, for i > 1, the equivalence continues. With this,

$$a_1x^{k+1} + a_2x^k + a_3x^{k-1} + a_4x^{k-2} + \cdots + + a_{k+2} =$$
$$a_1x^{k+1} + a_2x^k + a_3x^{k-1} + a_4x^{k-2} + \cdots + + a_{k+2} \quad (52)$$

Since the LHS = RHS, the general formula must be true for all n in the domain. To reiterate:

$$f(x) = \sum_{i=0}^{n} \frac{det(A_{B(i+1)})}{det(A)} x^{n-i} \quad (30)$$

Table 2. Parameters for the General Formula for an n[th] Degree Polynomial Sequence.

Symbol	Placeholder of
A	Matrix containing *x* values
B	Matrix containing *y* values
i	Lower limit of summation operator
n	Degree of polynomial, summation upper limit
x	Any value s.t. $x \in \mathbb{R}$

The derived formula provides a general way to solve for the desired polynomial equation satisfying given terms. However, it makes use of higher-level concepts such as matrices and determinants which may not be immediately intuitive when grasping for the first time.

In addition, it was also found that the formula computes for the polynomial function that best fits the outputs given, even if the degree of the produced polynomial equation is less than n. For instance, a sequence of 1, 4, 9, and 16, if solved using the given formula, will show $f(n) = n^2$, even if in theory, since there are four given terms, the degree of the equation should be 3, not 2.

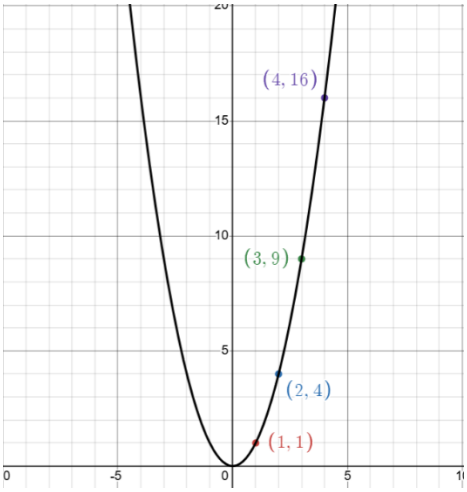

Figure 1. Graph of $f(n) = n^2$ with the four given outputs.

As such,

$$f(n) = n^2 \equiv 0n^3 + n^2 + 0n + 0 \quad (53)$$

In general, an n^{th} degree polynomial has *n+1* terms and consequently, coefficients. Matrix A is an *n+1* × *n+1* square matrix, and matrix B is a *n+1* × *1* matrix. This is to compensate for the constant term, where n = 0.

The formula functions according to Cramer's Rule. For practical uses, $\{n \in \mathbb{Z} | \, n \in (2,3)\}$ is recommended as it enables manageable calculation of determinants.

For computerized processes, such as PROPlexis, the general formula may be a faster alternative [9]. However, since the computational complexity of determinants by brute force is $O(n!)$, algorithms that can speed up this process such as LU decomposition and the Strassen algorithm, with computational complexities for systems of $O(n^3)$ and $\approx O(n^{log_2 7})$ respectively, are recommended [11].

B. Illustrative Example

To demonstrate how the formula works, an example is shown below.

Given 1, 8, and 29. Calculate for the fourth and fifth term.

1) Solution

Let n denote the input of a polynomial sequence $f(n)$, where the first three terms are given. The first step is to write the standard form of a polynomial sequence with degree that is at least one less than the number of given terms.

$$f(n) = a_1n^2 + a_2n + a_3 \quad (53)$$

After this, solve for the elements of matrix A and B by setting up a system of equations with the three different n values.

$$f(1) = 1 = a_1 + a_2 + a_3 \quad (54)$$

$$f(2) = 8 = 4a_1 + 2a_2 + a_3 \quad (55)$$

$$f(3) = 29 = 9a_1 + 3a_2 + a_3 \quad (56)$$

The LHS shall now be the elements of matrix B and the coefficients of the variables on the RHS shall be the elements of matrix A. Doing so gives the following:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 8 \\ 29 \end{bmatrix} \quad (57)$$

With all these now prepared, the general formula shall now be used. The process of solving for the coefficients are as follows:

$$f(n) = \sum_{i=0}^{2} \frac{det(A_{B(i+1)})}{det(A)} n^{2-i} \quad (58)$$

$$f(n) = \frac{det(A_{B1})}{det(A)} n^2 + \frac{det(A_{B2})}{det(A)} n + \frac{det(A_{B3})}{det(A)} \quad (59)$$

$$a_1 = \frac{det(A_{B1})}{det(A)} = \frac{det\begin{pmatrix} 1 & 1 & 1 \\ 8 & 2 & 1 \\ 29 & 3 & 1 \end{pmatrix}}{det\begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix}} \quad (60)$$

$$a_1 = \frac{1(2-3) - 1(8-29) + 1(24-58)}{1(2-3) - 1(4-9) + 1(12-18)} \quad (61)$$

$$a_1 = 7 \quad (62)$$

$$a_2 = \frac{det(A_{B2})}{det(A)} = \frac{det\begin{pmatrix}1 & 1 & 1\\ 4 & 8 & 1\\ 9 & 29 & 1\end{pmatrix}}{det\begin{pmatrix}1 & 1 & 1\\ 4 & 2 & 1\\ 9 & 3 & 1\end{pmatrix}} \quad (63)$$

$$a_2 = \frac{1(8-29) - 1(4-9) + 1(116-72)}{1(2-3) - 1(4-9) + 1(12-18)} \quad (64)$$

$$a_2 = -14 \quad (65)$$

$$a_3 = \frac{det(A_{B3})}{det(A)} = \frac{det\begin{pmatrix}1 & 1 & 1\\ 4 & 2 & 8\\ 9 & 3 & 29\end{pmatrix}}{det\begin{pmatrix}1 & 1 & 1\\ 4 & 2 & 1\\ 9 & 3 & 1\end{pmatrix}} \quad (66)$$

$$a_3 = \frac{1(58-24) - 1(116-72) + 1(12-18)}{1(2-3) - 1(4-9) + 1(12-18)} \quad (67)$$

$$a_3 = 8 \quad (68)$$

Therefore, the general equation of the sequence is $f(n) = 7n^2 - 14n + 8$. To solve for the fourth and fifth terms, substitute them into *n* and evaluate the function.

$$f(4) = 7(4)^2 - 14(4) + 8 = 64 \quad (69)$$

$$f(5) = 7(5)^2 - 14(5) + 8 = 113 \quad (70)$$

Thus, the fourth and fifth terms are 64 and 113 respectively.

Q.E.D.

IV. CONCLUSION

It has been shown that a general formula expressing the coefficients in terms of the outputs has been derived and validated using the principle of strong induction. The formula derived is as follows:

$$f(x) = \sum_{i=0}^{n} \frac{det\left(A_{B(i+1)}\right)}{det(A)} x^{n-i} \quad (30)$$

Table 2. Parameters for the General Formula for an n[th] Degree Polynomial Sequence.

Symbol	Placeholder of
A	Matrix containing the known x values
B	Matrix containing the y values
i	Lower limit of the summation operator, incremental
n	Degree of the polynomial, upper limit of the summation
x	Any value, $x \in \mathbb{R}$

For practical uses, $\{n \in \mathbb{Z} \mid n \in (2,3)\}$ is recommended, i.e., quadratic and cubic sequences, as it enables for the calculation of determinants that are only 3×3 and 4×4. However, for tasks that require higher-order polynomial solving, computational degree may be able to surpass n values > 3. With this, as the degree increases, other algorithms such as the Laplace Expansion, LU decomposition which specializes in dividing the matrix into lower and upper triangular matrices, and the Strassen Algorithm which divides larger matrices into smaller ones, are recommended to use as a complement to the general formula to make computation more efficient.

AUTHOR CONTRIBUTIONS

The sole author, Joseph D. Gonzales, wrote the entire paper along with the calculations and the proofing done to validate the results.

ACKNOWLEDGMENT

The author acknowledges his coaches, namely Mrs. Javine M. Tolledo and Dr. Jorge Victor M. Sales, for their expertise that has led to the success of this study. In addition, the author acknowledges Mrs. Vanessa Yvette V. Alberto, Mr. Jenzo B. Joson and Mr. Rion Faustus O. Benitez for their assistance and support that has helped solve the problems encountered in the study. Moreover, the author wholeheartedly acknowledges Ms. Alessandra P. Bohayo for the emotional support and assistance that has made this endeavor possible.

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